Comparison of Decisional Maps

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PLAN

Introduction

There are many interesting decision problems related with GIS

- Suitability for housing (*Joerin, 1997*)
- Railway corridor (*Mousseau and Chakhar, 2008*)
- Risk of degradation of a region (*Metchebon, 2010*)

$\Rightarrow$ map partitioned in geographic units (g.u.)
each g.u. assessed on an ordinal scale

= decisional map

After some time, the map has evolved

Example : Loulouka

Study of Loulouka's basin in Burkina Faso

Map representing the response to the risk of degradation

Geographic units : 229 squares 500m x 500m

Scale for response : 4 categories :
adequate, moderately adequate, weakly adequate, not adequate

Example of evolution

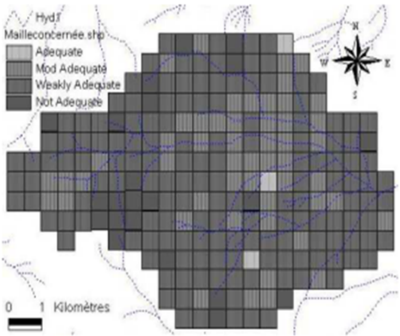

FIGURE 1 – Previous state

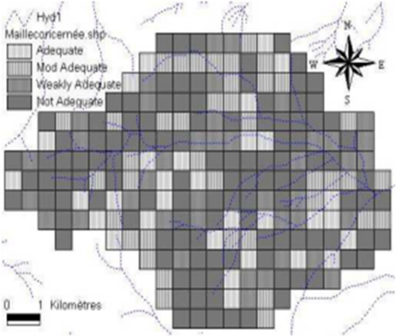

FIGURE 2 – New state

MODEL 1 :

Case of one decisional map

Model 1

Hypothesis

We assume that a map A is equivalent to the distribution of g.u. in categories :

$$x(A) = (x_1(A), x_2(A), \ldots, x_n(A))$$

Comparing maps is equivalent to comparing probability distributions

Model : EU (expected utility)

$$A \succsim B \iff \sum_i x_i(A)u_i \geq \sum_i x_i(B)u_i$$

where u_i is a value associated with the map in which all g.u. belong to category i

Model 1

Characterization : Adaptating Jensen's axioms (Fishburn 82)

- The set of maps is a mixture set : for $\lambda \in [0,1]$,

$$\lambda A \oplus (1-\lambda)B$$

  is a map in which a portion λ has the same distribution as A and a portion $1-\lambda$ has the same distribution as B
- $A = x_1(A)e_1 \oplus \ldots \oplus x_i(A)e_i \oplus \ldots \oplus x_n(A)e_n$ where e_i is a map with all g.u. in category i
- $\succsim$ is the DM's preference on the set of maps
- if this preference satisfies some axioms then it has a linear EU representation

Model 1

Axioms

A_1 : $\succsim$ is a weak order on the set of maps

A_2 : for any maps A, B, C and $\forall \lambda \in]0,1]$

$$A \succ B \Rightarrow \lambda A \oplus (1-\lambda)C \succ \lambda B \oplus (1-\lambda)C$$

A_3 : [Continuity] $A \succ B$ and $B \succ C$ imply
there exists a number $\alpha \in]0,1[$ such that $\alpha A \oplus (1-\alpha)C \succ B$
there exists a number $\beta \in]0,1[$ such that $B \succ \beta A \oplus (1-\beta)C$

Model 1

Representation theorem

There is a unique linear utility function u representing $\succsim$; i.e.

$$A \succsim B \iff u(A) \geq u(B)$$

Linear : $u(\lambda A \oplus (1-\lambda)B) = \lambda u(A) + (1-\lambda)u(B)$

Unique : if u' represents $\succsim$ then there are $a > 0$ and b such that $u' = au + b$

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Unique : if u' represents $\succsim$ then there are $a > 0$ and b such that $u' = au + b$

Benefits

$$A = x_1(A)e_1 \oplus \cdots \oplus x_i(A)e_i \oplus \cdots \oplus x_n(A)e_n$$

$$u(A) = x_1(A)u(e_1) + \cdots + x_i(A)u(e_i) + \cdots + x_n(A)u(e_n)$$

We only have to determine $u(e_i) := u_i$ for $i = 1, \ldots, n$

Model 1

Additional axiom

Due to the ordering of the categories, the DM's preference surely satisfies :

$$e_1 \succ e_2 \succ \cdots \succ e_n$$

which results in :

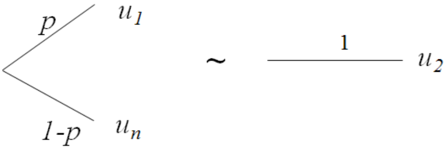
$$u_1 > u_2 > \cdots > u_n$$

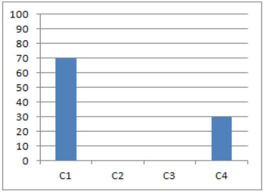
Using the two degrees of freedom, we set

$$u_1 = 100 \text{ and } u_n = 0$$

Elicitation

Inspired by comparison of lotteries





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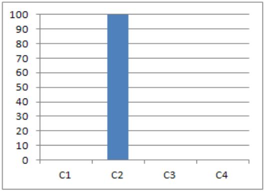


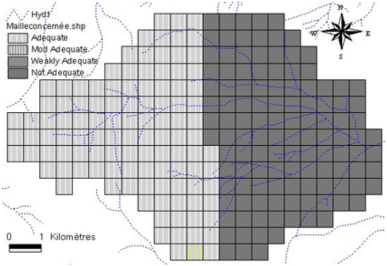
Remarks

Model 1 only takes into account the proportion of g.u. in the categories
$\rightarrow$ a bit disappointing : geographic aspects neglected

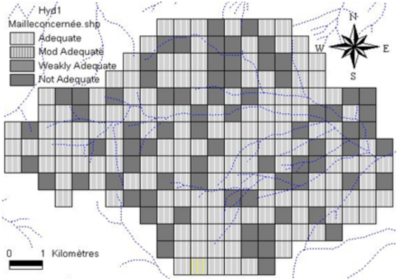
One geographic aspect : contiguity

Same distribution

Grouped



Scattered



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One geographic aspect : contiguity
$\rightarrow$ model based on the Choquet's integral w.r.t. a 2-additive capacity

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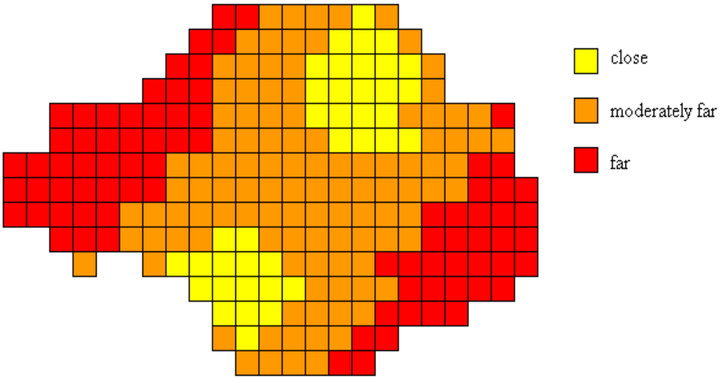
One geographic aspect : contiguity
$\rightarrow$ model based on the Choquet's integral w.r.t. a 2-additive capacity

Other geographic aspects : proximity of a village, of a road, of a watercourse, ...
$\rightarrow$ attribute map(s)

MODEL 2 :

Case of one decisional map
and
one attribute map

Fictitious attribute map



Model 2

We consider :

- a decisional map A
- a fixed attribute map G partitionned as A

We note (A, G) :

- a map partitionned in g.u.
- each g.u. assessed on 2 scales : the same as A and the same as G

Model 2

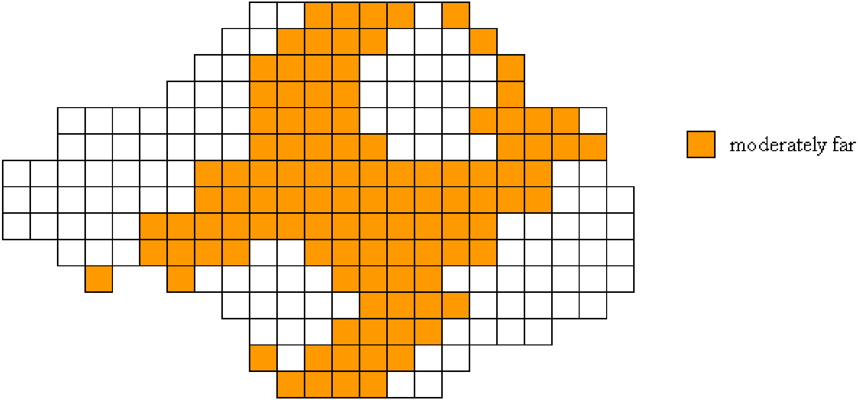
Hypothesis

We assume that a map (A, G) is equivalent to the distribution of g.u. in categories :

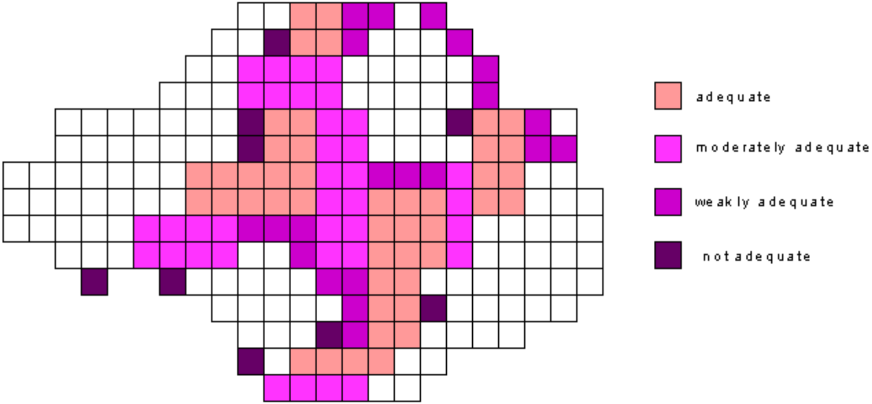
$$\begin{aligned} x(A, G) & \\ &= ((x_{11}(A,G),\ldots,x_{n1}(A,G)),\ldots,(x_{1m}(A,G),\ldots,x_{nm}(A,G))) \\ &:= (x_1(A,G),\ldots,x_m(A,G)) \end{aligned}$$

Comparing maps is equivalent to comparing vectors of probability distributions

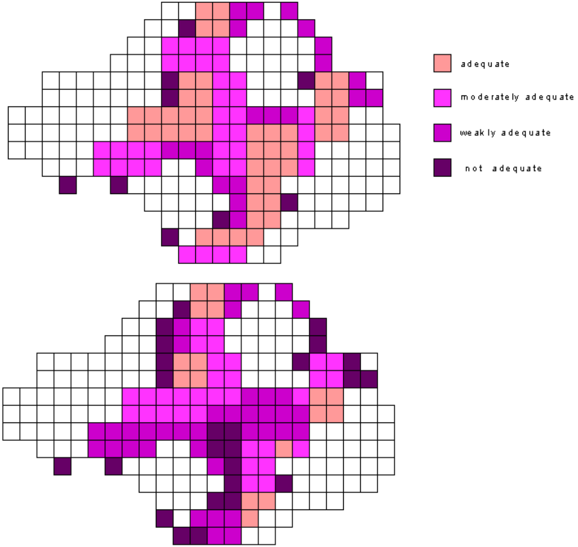
Example of partial map



Example of distribution in categories for a partial map



Preferences on partial maps



Model 2

Notation and definitions

- For $j \in \{1, \ldots, m\}$ we write X_j the set of all possible distributions in categories of a partial map
- For $j \in \{1, \ldots, m\}$ we write $X_{-j} = \prod_{\substack{k=1 \\ k \neq j}}^{m} X_k$
- For $x_j \in X_j$ and $a_{-j} \in X_{-j}$, we write

$$(x_j, a_{-j}) = (a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_m)$$

- We define a preference relation $\succsim_j$ on X_j as follows :

$$\forall x_j, y_j \in X_j, \;\; x_j \succsim_j y_j \iff \forall a_{-j} \in X_{-j} \;\; (x_j, a_{-j}) \succsim (y_j, a_{-j})$$

Model 2

Notations and definitions (following)

- Each set X_j is a mixture set : for $\lambda \in [0, 1]$

$$\lambda x_j \oplus (1 - \lambda) y_j$$

  is a partial map in which a portion λ has the same distribution as x_j, the other portion $1 - \lambda$ having the same distribution as y_j

Model 2

Axioms

B_1 : $\succsim$ is a weak order

B_2 : $\forall j \in \{1, \ldots, m\}$ $\forall x_j, y_j, z_j \in X_j$ $\forall a_{-j} \in X_{-j}$ and $\forall \lambda \in]0, 1]$

$$[(x_j, a_{-j}) \succ (y_j, a_{-j})] \Rightarrow$$
$$[\lambda(x_j, a_{-j}) \oplus (1 - \lambda)(z_j, a_{-j}) \succ \lambda(y_j, a_{-j}) \oplus (1 - \lambda)(z_j, a_{-j})]$$

B_3 : $\forall j \in \{1, \ldots, m\}$ $\forall x_j, y_j, z_j \in X_j$ $\forall a_{-j} \in X_{-j}$

$$[(x_j, a_{-j}) \succ (y_j, a_{-j}) \succ (z_j, a_{-j})] \Rightarrow \exists \alpha, \beta \in]0, 1[:$$
$$\alpha(x_j, a_{-j}) \oplus (1 - \alpha)(z_j, a_{-j}) \succ (y_j, a_{-j})$$

and

$$(y_j, a_{-j}) \succ \beta(x_j, a_{-j}) \oplus (1 - \beta)(z_j, a_{-j})$$

Model 2

Axioms (following)

B_4 [Essentiality] : $\forall j \in \{1, \ldots, m\}\ \exists x_j, y_j \in X_j\ \exists a_{-j} \in X_{-j}$:

$$(x_j, a_{-j}) \succ (y_j, a_{-j})$$

B_5 [Independence] : $\forall j \in \{1, \ldots, m\}\ \forall x_j, y_j \in X_j\ \forall a_{-j}, b_{-j} \in X_{-j}$

$$(x_j, a_{-j}) \succsim (y_j, a_{-j}) \Rightarrow (x_j, b_{-j}) \succsim (y_j, b_{-j})$$

B_6 [Restricted solvability] :
$\forall j \in \{1, \ldots, m\}\ \forall x_j, z_j \in X_j\ \forall a_{-j} \in X_{-j}\ \forall y \in X$

$$(x_j, a_{-j}) \succsim y \succsim (z_j, a_{-j}) \Rightarrow \exists w_j \in X_j : y \sim (w_j, a_{-j})$$

B_7 [Archimedean] : $\forall j \in \{1, \ldots, m\}$, if a standard sequence on X_j is bounded, then it is finite

Model 2

Representation theorem

- $\succsim$ is reprented by an additive value function

$$(A, G) \succsim (B, G) \iff \sum_{j=1}^{m} u_j(x_j(A, G)) \geq \sum_{j=1}^{m} u_j(x_j(B, G))$$

- $\succsim_j$ is represented by a linear EU function

$$x_j(A, G) = x_{1j}(A, G)e_{1j} \oplus \cdots \oplus x_{nj}(A, G)e_{nj}$$

where e_{ij} is a partial map where all g.u. are in i and j

$$u_j(x_j(A, G)) = \sum_{i=1}^{n} x_{ij}(A, G)u_j(e_{ij})$$

Model 2

Representation theorem

$$(A, G) \succsim (B, G) \iff \sum_j \sum_i x_{ij}(A, G) u_j(e_{ij}) \geq \sum_j \sum_i x_{ij}(B, G) u_j(e_{ij})$$

We have to determine $u_j(e_{ij})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$

Model 2

Additional axiom

Due to the ordering of the categories, the DM's preference surely satisfies, for any j :

$$e_{1j} \succ e_{2j} \succ \cdots \succ e_{nj}$$

which results in :

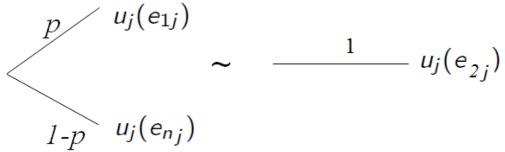
$$u_j(e_{1j}) \succ u_j(e_{2j}) \succ \cdots \succ u_j(e_{nj})$$

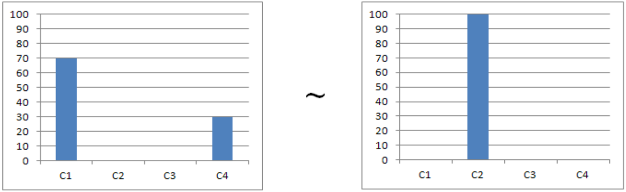
Using the two degrees of freedom, we set

$$u_j(e_{1j}) = 100 \text{ and } u_j(e_{nj}) = 0$$

Elicitation

Inspired by comparison of lotteries



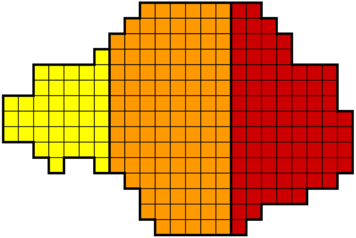


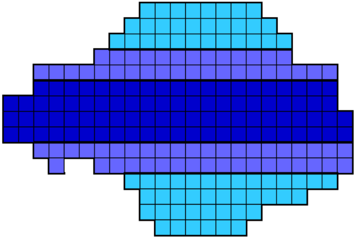
MODEL 3 :

Case of one decisional map and several attribute maps

Idea : create a unique attribute map

- each g.u. assessed on $l + 1$ scales (if l attribute maps)





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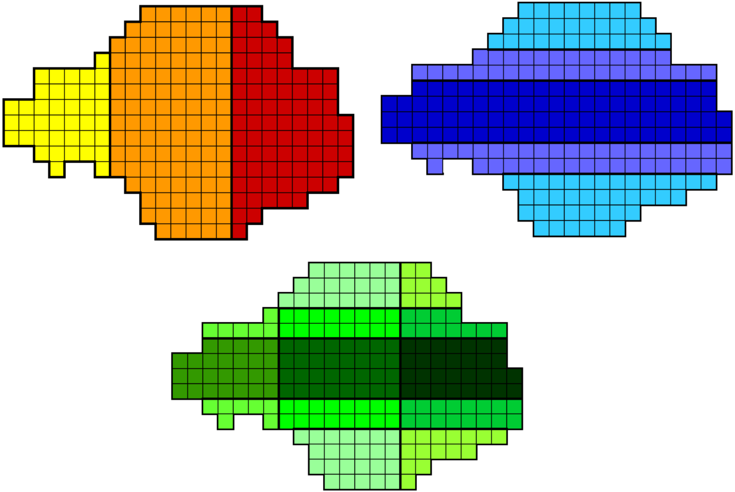

Model 3

Model : EU (expected utility)

$$(A, G_1, \dots, G_l) \succsim (B, G_1, \dots, G_l) \iff \sum_{j_1} \cdots \sum_{j_l} \sum_i x_{i,j_1,\dots,j_l}(A, G_1, \dots, G_l) u_{j_1,\dots,j_l}(e_{i,j_1,\dots,j_l}) \geq \sum_{j_1} \cdots \sum_{j_l} \sum_i x_{i,j_1,\dots,j_l}(B, G_1, \dots, G_l) u_{j_1,\dots,j_l}(e_{i,j_1,\dots,j_l})$$

where $u_{j_1,\dots,j_l}(e_{i,j_1,\dots,j_l})$ is a value associated with the map in which all g.u. belong to category i and characteristic j_k on G_k, $k = 1, \dots, l$

Conclusion

Emphasize the usefulness of characterizations $\rightarrow$ elicitation
Much work still to be done
For instance :

- implement a decision deck tool for questioning the DM in terms of histograms comparisons or maps comparisons
- include contiguity in models 2 and 3
- reduce the number of parameters to be elicited $\rightarrow$ other characterization
- outranking methods